ELSEVIER

Journal of Pure and Applied Algebra 138 (1999) 239–249

JOURNAL OF PURE AND APPLIED ALGEBRA

Properties of AS-Cohen–Macaulay algebras

Peter Jørgensen

Matematisk Afdeling, Københavns Universitet, Universitetsparken 5, DK-2100 København Ø, Denmark

Communicated by C.A. Weibel; received 5 March 1997

Abstract

An AS-Cohen–Macaulay algebra is the non-commutative graded analogue of a (commutative local) Cohen–Macaulay ring. This note will show how some central properties of commutative Cohen–Macaulay rings generalize to AS-Cohen–Macaulay algebras. We prove the following results.

Theorem. *An AS-Cohen–Macaulay algebra has a balanced dualizing complex if and only if it is a graded factor of an AS-Gorenstein algebra.*

Theorem. *Let A be an FBN AS-Cohen–Macaulay algebra. Then*
- *A has an artinian ring of quotients.*
- *Every minimal prime ideal $\mathfrak{p}$ of A is graded, and*

$$\operatorname{GK}\dim(A/\mathfrak{p}) = \operatorname{GK}\dim(A).$$

- *A has a balanced dualizing complex of the form $K[n]$ for a bi-module K, and if $x \in A$ is regular, then x is also regular on K (from both sides).*

Theorem. *Let A be FBN, $\mathbb{N}$-graded and connected. Then the following conditions are equivalent*:
- *The algebra A is AS-Cohen–Macaulay.*
- *We have* $\operatorname{depth}_A(A) = \operatorname{GK}\dim(A)$.
- *The algebra A satisfies the following "inequality of the grade": for any $X \in D^{\mathrm{b}}_{\mathrm{fg}}(\operatorname{GrMod}(A))$, we have the inequality*

$$\operatorname{grade}_A(X) \geq \operatorname{GK}\dim(A) - \operatorname{GK}\dim(X).$$

AMS Classification: 16W50; 16E30

0022-4049/99/$ – see front matter © 1999 Elsevier Science B.V. All rights reserved.
PII: S0022-4049(98)00086-3

0. Introduction

It is classical that, on the one hand, the theory of commutative noetherian local rings, and, on the other hand, the theory of commutative noetherian graded algebras have many similarities. And in fact, as has been realized more recently, the similarities extend to the theory of non-commutative noetherian graded algebras. For instance, one can construct a theory of dualizing complexes and local cohomology in the non-commutative graded case, see [4, 9, 10]. It is also possible to make non-commutative graded analogues of the notions of (commutative) regular, Gorenstein and Cohen–Macaulay rings, and several results concerning regular and Gorenstein rings have been generalized from the commutative local to the non-commutative graded situation, see [8, 12].

The purpose of this note is to perform similar generalizations of some results about Cohen–Macaulay rings. Their non-commutative graded analogues are called AS-Cohen–Macaulay algebras, and are defined in [9, p. 10] and below, in Definition 1.1. The central result is proved in Section 1: an AS-Cohen–Macaulay algebra has a (balanced) dualizing complex if and only if it is a graded factor of an AS-Gorenstein algebra. The corresponding result for commutative local rings is well known, and proved independently in [2, 6].

Since a lot is known about AS-Gorenstein algebras, particularly when they are FBN, we can use the result from Section 1 to derive some properties of FBN AS-Cohen–Macaulay algebras. This is done in Section 2, which shows that a number of properties, previously shown for FBN AS-Gorenstein algebras in [8, 12], concerning prime ideals and rings of quotients, are in fact valid for FBN AS-Cohen–Macaulay algebras; see Theorem 2.1. Section 2 also gives some alternative ways of characterizing AS-Cohen–Macaulay algebras.

A thing to be noted is the fact that Section 1 is constructive: given an AS-Cohen–Macaulay algebra, A, with a balanced dualizing complex, Section 1 constructs a concrete AS-Gorenstein algebra, B, which has A as graded factor. The method of proof can therefore be viewed as a "factory" for producing new examples of AS-Gorenstein algebras. In fact, B is constructed by the same prescription as in the commutative case, cf. [2, 6]: A's dualizing complex is given by a single graded bi-module K, and B is obtained as the "idealization", $A \times K(-s)$, which contains $K(-s)$ as an ideal.

A practical remark: the mathematical notation employed below is explained in detail in [4], and is very close to the notation in [10]. These two references can also be consulted for an explanation of the methods of derived categories in the theory of non-commutative rings.

1. Existence of dualizing complexes

In this section it will be proved that an AS-Cohen–Macaulay algebra A has a balanced dualizing complex in the sense of [10, Definitions 3.3 and 4.1], if and only if A is graded factor of an AS-Gorenstein algebra.

First let us restate the definition of AS-Cohen–Macaulay algebras.

Definition 1.1. We call A an AS-Cohen–Macaulay algebra when A is a noetherian, $\mathbb{N}$-graded, connected k-algebra, for which the complex $R\Gamma_{\mathfrak{m}}(A)$ is concentrated in one degree.

If a noetherian $\mathbb{N}$-graded connected k-algebra A admits a balanced dualizing complex, it is given by $R\Gamma_{\mathfrak{m}}(A)'$ (see [9, Theorem 6.3]). When such an A is AS-Cohen–Macaulay, we can write this complex as $R\Gamma_{\mathfrak{m}}(A)' \cong K[n]$ for a certain graded A-bi-module K. In this case, we call K the dualizing module for A.

Now a few preliminaries concerning the process of idealization.

Definition 1.2. Let R be a ring, B an R-bi-module. Then we equip the product $R \times B$ with a multiplication by setting

$$(r,b) \cdot (r_1,b_1) = (rr_1, rb_1 + br_1).$$

Lemma 1.3. *Let R be a ring, B an R-bi-module. Then $R \times B$ is again a ring, and the maps*

$$R \to R \times B, \quad r \mapsto (r,0)$$

and

$$R \times B \to R, \quad (r,b) \mapsto r$$

are ring homomorphisms.

Proof. This can be checked directly. □

Note that the ring $R \times B$ contains B as an ideal (namely as the ideal $\{(0,b) \mid b \in B\}$), whence the term "idealization".

Proposition 1.4. *Let A be a noetherian, $\mathbb{N}$-graded, connected k-algebra, and let K be a graded A-bi-module, finitely generated from both sides. Put $B = A \times K(-s)$, where s is strictly larger than $-\mathrm{i}(K)$ (by $\mathrm{i}(K)$, we mean the degree of the lowest non-vanishing graded piece of K). We shall write $\mathfrak{m} = A_{\geq 1}$ and $\mathfrak{n} = B_{\geq 1}$. Then*

(1) *B is a noetherian, $\mathbb{N}$-graded, connected k-algebra.*

(2) *The ungraded prime ideals of B are exactly the sets of the form $\mathfrak{p} \times K(-s)$, where $\mathfrak{p}$ is an ungraded prime ideal of A. Moreover, $\mathfrak{p} \times K(-s)$ is a graded ideal precisely when $\mathfrak{p}$ is. The nil radical of B is given by $N_B = N_A \times K(-s)$.*

(3) *If A and A^{opp} satisfy the condition χ, then so do B and B^{opp}.*

(4) *Suppose that A and A^{opp} satisfy χ. If $\mathrm{lcd}(A) < \infty$ and $\mathrm{lcd}(A^{\mathrm{opp}}) < \infty$, then also $\mathrm{lcd}(B) < \infty$ and $\mathrm{lcd}(B^{\mathrm{opp}}) < \infty$ (by $\mathrm{lcd}(A)$, we mean the cohomological dimension of the "local section functor" $\Gamma_{\mathfrak{m}}$ on the category $\mathrm{GrMod}(A)$).*

Proof. (1) It is clear how B acquires its grading from the grading of A and K. It is also clear that when $s > -\mathrm{i}(K)$, then $K(-s)$ is translated so far to the right that it only affects positive degrees, whence B becomes $\mathbb{N}$-graded and connected.

To see that B is noetherian, we note from Lemma 1.3 that we have the homomorphism of graded algebras,

$$\alpha : A \to B, \quad \alpha(a) = (a, 0),$$

so B is a (graded) module over A. And since K is finitely generated from both sides, B is also finitely generated over A from both sides. Since A is noetherian, we see that B is noetherian.

(2) Let P be an ungraded prime-ideal in B. If $k \in K(-s)$, we can compute inside B as follows:

$$(0,k)B(0,k) = (0,k) \cdot \{(0, ak) \mid a \in A\} = \{(0,0)\} \subseteq P,$$

and so we learn that $(0,k) \in P$. Since k is arbitrary, this shows that P must have the form

$$P = \mathfrak{p} \times K(-s)$$

for some subset $\mathfrak{p} \subseteq A$. It is easy to see that the $\mathfrak{p}$ in question must be an ideal, and that $B/P \cong A/\mathfrak{p}$. So since P is prime, so is $\mathfrak{p}$.

Conversely, when $\mathfrak{p}$ is an ungraded prime ideal in A, we look at $P = \mathfrak{p} \times K(-s)$ in B. Again we can check that P is an ideal, and that $B/P \cong A/\mathfrak{p}$. So since $\mathfrak{p}$ is prime, so is P.

It is clear that the ideal $\mathfrak{p} \times K(-s)$ is graded if and only if the ideal $\mathfrak{p}$ is graded.

Finally, we have

$$N_B = \bigcap_{P \in \operatorname{Spec}(B)} P = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(A)} \mathfrak{p} \times K(-s) = \left(\bigcap_{\mathfrak{p} \in \operatorname{Spec}(A)} \mathfrak{p} \right) \times K(-s) = N_A \times K(-s).$$

(3) Both A and B are noetherian, and the homomorphism of graded algebras α makes B into an A-module which is finitely generated from both sides. But then χ for A implies χ for B, by [1, Theorem 8.3(2)]. The same argument works for the opposite algebras.

(4) Since A and A^{opp} satisfy χ, we know from (3) that B and B^{opp} also satisfy this condition. And when B satisfies χ, we know from [1, Theorem 8.3(3)] that if M is a finitely generated graded B-module, and $i \geq 2$ an integer, we have $H^i_{\mathfrak{n}}(M) \cong H^i_{\mathfrak{m}}({}_A M)$. Since local cohomology commutes with small filtering direct limits (see [9, Lemma 4.3]), this proves that $\operatorname{lcd}(B) \leq \operatorname{lcd}(A)$. The same proof works for the opposite algebras. $\square$

Now we show that, just as in the commutative case, if we idealize the dualizing module of an AS-Cohen–Macaulay algebra, we get an AS-Gorenstein algebra.

Proposition 1.5. *Let A be an AS-Cohen–Macaulay algebra which admits a balanced dualizing complex, $K[n]$, where K is A's dualizing module. Put $B = A \times K(-s)$, where s is strictly larger than $-\mathrm{i}(K)$.*

Then B is AS-Gorenstein, with $\mathrm{id}_B(B) = \mathrm{id}_{B^{\mathrm{opp}}}(B) = n$ *and*

$$\underline{\mathrm{Ext}}^i_B(k,B) = \underline{\mathrm{Ext}}^i_{B^{\mathrm{opp}}}(k,B) = \begin{cases} 0 & \text{for } i \neq n, \\ k(-s) & \text{for } i = n. \end{cases}$$

Proof. *Part* I: because of [9, Theorem 6.3], the algebras A and A^{opp} satisfy χ, and both functors $\Gamma_{\mathfrak{m}}$ and $\Gamma_{\mathfrak{m}^{\mathrm{opp}}}$ have finite cohomological dimension. This means that all four parts of Proposition 1.4 apply to the case at hand. In particular, B will satisfy the conditions of [9, Theorem 6.3], and thus has a balanced dualizing complex.

From Lemma 1.3, we have the homomorphism of graded algebras

$$\beta : B \to A, \quad \beta((a,k)) = a,$$

which presents A as $B/\mathrm{Ker}(\beta)$. We can also use it to view A as a graded B-bi-module, and we obtain a short exact sequence of graded B-bi-modules,

$$0 \to K(-s) \to B \xrightarrow{\beta} A \to 0, \tag{1}$$

where by $K(-s)$ we mean the ideal

$$\{(0,k) \mid k \in K(-s)\} \subseteq B.$$

The two outer modules in the sequence (1) can be viewed as A-bi-modules, since they are annihilated from both sides by $\mathrm{Ker}(\beta)$, and as A-bi-modules, they really are $K(-s)$, respectively, A.

We can take the long exact sequence for the functors $R^i\Gamma_{\mathfrak{n}}$ applied to the short exact sequence (1). This gives a sequence of graded B-bi-modules. And because of [4, Lemma 3.1], if X is a graded A-bi-module, then for each i, we have $R^i\Gamma_{\mathfrak{n}}(X) \cong R^i\Gamma_{\mathfrak{m}}(X)$ as B-bi-modules. So our long exact sequence consists of pieces

$$R^i\Gamma_{\mathfrak{m}}(K)(-s) \to R^i\Gamma_{\mathfrak{n}}(B) \to R^i\Gamma_{\mathfrak{m}}(A),$$

and [10, Definition 4.1; 9, Theorem 6.3] then make it clear that among the $R^i\Gamma_{\mathfrak{n}}(B)$, only number n is different from zero, and sits in a short exact sequence

$$0 \to R^n\Gamma_{\mathfrak{m}}(K)(-s) \to R^n\Gamma_{\mathfrak{n}}(B) \to R^n\Gamma_{\mathfrak{m}}(A) \to 0. \tag{2}$$

Now, the balanced dualizing complex for B is $R\Gamma_{\mathfrak{n}}(B)'$, so it is clear that B is AS-Cohen–Macaulay. If we write $L = R^n\Gamma_{\mathfrak{n}}(B)'$, the balanced dualizing complex for B is $L[n]$. And due to [10, Definition 4.1; 9, Theorem 6.3], if we take Matlis-dual of the short-exact sequence (2), we get

$$0 \to K \to L \to A(s) \to 0,$$

so shifting, we have the following short exact sequence of graded B-bi-modules:

$$0 \to K(-s) \xrightarrow{\kappa} L(-s) \xrightarrow{\lambda} A \to 0. \tag{3}$$

In particular, this proves that $L(-s)$ has the same Hilbert series as B itself,

$$H_{L(-s)}(t) = H_B(t).$$

Part II: There is an element $e \in L(-s)_0$ such that $\lambda(e) = 1_A$. Suppose now that b is a homogeneous element in the set

$$\mathrm{leftann}_B(e) = \{x \in B \mid xe = 0\}.$$

Write $b = (a, k)$ with $a \in A$ and $k \in K(-s)$. We see that

$$0 = \lambda(0) = \lambda(be) = b\lambda(e) = (a,k)1_A = a$$

(by $(a,k)1_A$, we mean the A-element 1_A multiplied by the B-scalar (a,k)), so in fact $b = (0, k)$.

Let $l \in L(-s)$ be some element. If we write $\lambda(l) = a_1 \in A$, we have $\lambda(l) = 1_A(a_1, 0)$. So

$$\lambda(l - e(a_1,0)) = \lambda(l) - \lambda(e)(a_1,0) = \lambda(l) - 1_A(a_1,0) = 1_A(a_1,0) - 1_A(a_1,0) = 0,$$

and since the sequence (3) is exact, we can find $k_1 \in K(-s)$ such that $l - e(a_1,0) = \kappa(k_1)$. However, then

$$bl = b(e(a_1,0) + \kappa(k_1)) = (be)(a_1,0) + \kappa(bk_1) = 0 + \kappa((0,k)k_1) = 0 + 0 = 0,$$

but this can be performed for any l, so b annihilates all of $L(-s)$ from the left.

On the other hand, $L[n]$ is B's balanced dualizing complex, so from [10, Definition 3.3(iii)], it is clear that the canonical map $B \to \underline{\mathrm{Hom}}_{B^{\mathrm{opp}}}(L(-s), L(-s))$ is an isomorphism. So if $b \in B$ were non-zero, the map

$$L(-s) \to L(-s + \deg(b)), \quad l \mapsto bl$$

would be non-zero, that is, b could not annihilate $L(-s)$ from the left. All in all, we see that $b = 0$. So

$$\mathrm{leftann}_B(e) = 0,$$

and together with the aforementioned fact $H_{L(-s)}(t) = H_B(t)$, this shows that if we consider $L(-s)$ as a B-left-module, it is free, with e as generator.

The same argument can be used from the right, and shows that as a B-right-module, $L(-s)$ is also free, with e as generator.

Part III: To finish the proof, we let $M \in \mathrm{GrMod}(B)$ be a left-module, and write up the local duality formula from [9, Theorem 5.1]:

$$R\Gamma_{\mathfrak{m}}(M)' \cong R\,\underline{\mathrm{Hom}}_B(M, L[n]) \cong R\,\underline{\mathrm{Hom}}_B(M, B(s)[n]),$$

since as left-modules, $L \cong B(s)$. This means

$$R^i\Gamma_{\mathfrak{m}}(M)' \cong \underline{\mathrm{Ext}}_B^{n-i}(M, B)(s).$$

There is a corresponding formula from the right:

$$R^i\Gamma_{\mathfrak{m}^{\mathrm{opp}}}(M)' \cong \underline{\mathrm{Ext}}_{B^{\mathrm{opp}}}^{n-i}(M, B)(s).$$

But these two formulae clearly imply the proposition's statements about the left- and right-injective dimensions of B and about the modules

$$\underline{\mathrm{Ext}}^i_B(k,B) \quad \text{and} \quad \underline{\mathrm{Ext}}^i_{B^{\mathrm{opp}}}(k,B). \qquad \square$$

The algebra A is graded factor of the idealization $A \times K(-s)$, so as consequence of Proposition 1.5, we have that an AS-Cohen–Macaulay algebra with a dualizing complex is graded factor of an AS-Gorenstein algebra. The converse statement is also true. We sum this up in the following generalization of the commutative result from [2, 6]:

Theorem 1.6. *Let A be an AS-Cohen–Macaulay algebra. Then A has a balanced dualizing complex if and only if there exists an AS-Gorenstein algebra B, with a graded ideal $\mathfrak{b}$, such that $A \cong B/\mathfrak{b}$.*

Proof. Assume that A admits the balanced dualizing complex $R\Gamma_{\mathfrak{m}}(A)' = K[n]$. Letting $s > -\mathrm{i}(K)$, we are in the situation of Proposition 1.5, so the algebra $A \times K(-s)$ is AS-Gorenstein. And from the proof of the proposition, we have the surjection of graded algebras

$$\beta : A \times K(-s) \to A,$$

whence A has been obtained as graded factor of an AS-Gorenstein algebra.

Next assume that $A \cong B/\mathfrak{b}$, where B is AS-Gorenstein, and $\mathfrak{b}$ a graded ideal. We know from [11, Corollary 4.3] that B satisfies χ and that $\mathrm{lcd}(B) < \infty$, and the same statements hold for B^{opp}, since the AS-Gorenstein condition is left/right-symmetric. So by [1, Theorem 8.3], the same statements are true for A, and by [9, Theorem 6.3] this means that A admits a balanced dualizing complex. $\square$

On the basis of this, one could be tempted to conjecture, as Sharp did in the commutative case in [7, Conjecture 4.4], that the only algebras possessing balanced dualizing complexes are the graded factors of AS-Gorenstein algebras.

2. FBN AS-Cohen–Macaulay algebras

This section uses the results of the previous section to show some properties of AS-Cohen–Macaulay algebras which are FBN. The idea is to note that if A is FBN, then the idealization $A \times K(-s)$ is also FBN, and then use what is already known about the behaviour of FBN AS-Gorenstein algebras, see [8, 12]. By these means, it is possible to derive Theorem 2.1, containing a number of ring theoretical properties of FBN AS-Cohen–Macaulay algebras which generalize known results on FBN AS-Gorenstein algebras. One can also prove Theorem 2.2, giving some alternative characterizations of FBN AS-Cohen–Macaulay algebras.

In the following theorem, note that parts (2)–(4) are direct generalizations of corresponding results for commutative noetherian local rings. In the commutative case, property (3) is a special case of [5, Theorem 17.3(i)]. Property (4) also follows from [5, Theorem 17.3(i)] by noting that the only associated prime ideals of the dualizing module, K, are the minimal prime ideals of A, whence the only zero-divisors on K are the zero-divisors of A itself. I have not been able to find a direct statement of property (2) in the commutative case, perhaps because the existence of artinian rings of quotients is not traditionally studied in commutative algebra. But the commutative statement can be proved by combining [5, Theorem 17.3(i)] with [3, Theorem 10.13].

Theorem 2.1. *Let A be an FBN AS-Cohen–Macaulay algebra. Then*

(1) *The algebra A admits a balanced dualizing complex.*

(2) *A has an artinian ring of quotients.*

(3) *Every minimal prime ideal $\mathfrak{p}$ of A is graded, and*

$$\mathrm{GK}\dim(A/\mathfrak{p}) = \mathrm{K}\dim(A/\mathfrak{p}) = \mathrm{GK}\dim(A).$$

(4) *If K is A's dualizing module, and $x \in A$ is a non-zero-divisor of A, then x is regular on K, both from the left and from the right.*

Proof. (1) Since A is FBN, [8, Lemma 6.1] applies, so in particular, $\mathrm{K}\dim({}_AA) < \infty$. By [1, Theorem 8.13], the algebra A satisfies χ, and $\mathrm{lcd}(A) \le \mathrm{K}\dim({}_AA) < \infty$; by the mirror version of the argument, the same things hold for A^{opp}. By [9, Theorem 6.3], the algebra A therefore admits a balanced dualizing complex. We shall write it as $R\Gamma_{\mathfrak{m}}(A)' = K[n]$.

To prepare for the rest of the proof, note from the proof of Theorem 1.6 that if $s > -\mathrm{i}(K)$, then $B = A \times K(-s)$ is AS-Gorenstein, and the map

$$\beta : B \to A, \quad (a,k) \mapsto a$$

exhibits A as graded factor of B. From Proposition 1.4(2), we know that the prime factors of B are in fact the same as the prime factors of A, so B is FBN because A is. It follows from [8, Theorem 6.2] and its proof that B is Auslander-Gorenstein and graded Cohen–Macaulay, the latter term used in the sense of [8, p. 989] to mean that

$$\mathrm{grade}_B(M) + \mathrm{GK}\dim(M) = \mathrm{GK}\dim(B)$$

for any finitely generated graded module M. We can use [8, Theorem 6.2] because B is "graded injectively smooth" by [8, Lemma 3.12]. So [12, Theorem 3.2] (which only assumes graded Cohen–Macaulayness, in the sense of [8, p. 989]) applies, and will be used below:

(2) By [12, Theorem 3.2(1)], the algebra B has an artinian ring of quotients. By [3, Theorem 10.10], this is equivalent to $\mathscr{C}_B(0) = \mathscr{C}_B(N_B)$. We would like to see the same for A, that is, $\mathscr{C}_A(0) = \mathscr{C}_A(N_A)$. The inclusion "$\subseteq$" can be found in [3, Lemma 10.8].

To see the reverse inclusion, let $a \in \mathscr{C}_A(N_A)$. This implies that when $c \in A \backslash N_A$, then $ac \notin N_A$. And if $(c,k) \in B \backslash N_B$, then $c \notin N_A$ (since $N_B = N_A \times K(-s)$), whence

$$(a,0)(c,k) = (ac, ak) \notin N_B.$$

By a similar argument, $(c,k)(a,0)=(ca,ka)\notin N_B$, proving

$$(a,0)\in\mathscr{C}_B(N_B)=\mathscr{C}_B(0).$$

But let $c\in A\setminus\{0\}$. Then since $(a,0)$ is regular in B, we must have

$$(a,0)(c,0)=(ac,0)\neq 0,$$

so $ac\neq 0$. Similarly, $ca\neq 0$. So a is regular in A, that is, $a\in\mathscr{C}_A(0)$, as required.

Note that along the way, we saw that if $a\in\mathscr{C}_A(N_A)$, then $(a,0)\in\mathscr{C}_B(N_B)$. Since all elements of the form $(0,k)$ are in N_B, we see that even $(a,k)\in\mathscr{C}_B(N_B)$, whence

$$\mathscr{C}_A(N_A)\times K(-s)\subseteq\mathscr{C}_B(N_B),$$

and knowing, as we now do, that $\mathscr{C}(0)=\mathscr{C}(N)$ for both A and B, we can rewrite this as

$$\mathscr{C}_A(0)\times K(-s)\subseteq\mathscr{C}_B(0).$$

(3) By the proof of [12, Theorem 3.3], the fact that A is connected graded ensures that all minimal prime ideals of A are graded. And if $\mathfrak{p}$ is a minimal prime ideal of A, then by Proposition 1.4(2), the ideal $\mathfrak{p}\times K(-s)$ is a minimal prime ideal of $B=A\times K(-s)$, and since $A/\mathfrak{p}\cong A\times K(-s)/\mathfrak{p}\times K(-s)$, we have

$$\operatorname{GK}\dim(A/\mathfrak{p})=\operatorname{GK}\dim\left(\frac{A\times K(-s)}{\mathfrak{p}\times K(-s)}\right)=\operatorname{GK}\dim(A\times K(-s))=\operatorname{GK}\dim(A),$$

where the second "=" uses [12, Theorem 3.2(2)]. Finally by [8, Lemma 6.1], GK dim and K dim coincide for finitely generated graded A-modules, so $\operatorname{K}\dim(A/\mathfrak{p})$ is equal to $\operatorname{GK}\dim(A/\mathfrak{p})$.

(4) Let $x\in A$ be a non-zero-divisor, i.e., $x\in\mathscr{C}_A(0)$. By the final remark of the proof of part (2) above, this implies $(x,0)\in\mathscr{C}_B(0)$. But then if $k\in K(-s)\setminus\{0\}$, we must have

$$(x,0)(0,k)=(0,xk)\neq 0,$$

that is, $xk\neq 0$. So x is left-regular on K. The same argument works from the right. $\square$

For commutative noetherian local rings, the following different characterizations of the Cohen–Macaulay property are well-known.

Theorem 2.2. *Let A be an FBN, $\mathbb{N}$-graded, connected k-algebra. Then the following conditions are equivalent*:

(1) *The algebra A is AS-Cohen–Macaulay.*

(2) *We have* $\operatorname{depth}_A(A)=\operatorname{GK}\dim(A)$.

(3) *The algebra A satisfies the following "inequality of the grade": for any $X\in D^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{GrMod}(A))$, we have the inequality*

$$\operatorname{grade}_A(X)\geq\operatorname{GK}\dim(A)-\operatorname{GK}\dim(X)$$

(*see* [4, Definition 5.1] *for the notion of* GK dim *of a complex of modules. When applied to a complex which is concentrated in degree zero, it coincides with the usual* GK dim).

Proof. First note that by the same argument as in the proof of Theorem 2.1(1), the algebra A has a balanced dualizing complex.

(1) $\Rightarrow$ (2): As noted in the proof of Theorem 2.1, when the FBN ring A is AS-Cohen–Macaulay, it is graded factor of an FBN AS-Gorenstein algebra B. Such an algebra satisfies the Similar Submodule Condition of [12], so we know from [4, Theorem 5.2] that for any $X \in D^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{GrMod}(A))$, we have $\sup R\Gamma_{\mathfrak{m}}(X) = \mathrm{GK}\dim(X)$. So since $R\Gamma_{\mathfrak{m}}(A)$ is concentrated in one degree,

$$\mathrm{depth}_A(A) = \inf R\Gamma_{\mathfrak{m}}(A) = \sup R\Gamma_{\mathfrak{m}}(A) = \mathrm{GK}\dim(A).$$

(1) $\Rightarrow$ (3): By using [10, Lemma 4.19] for the second "=" in the following computation, we get

$$\begin{aligned}
\mathrm{grade}_A(X) &= \inf R\,\underline{\mathrm{Hom}}_A(X, A) \\
&= \inf R\,\underline{\mathrm{Hom}}_{A^{\mathrm{opp}}}(R\Gamma_{\mathfrak{m}}(A)', R\Gamma_{\mathfrak{m}}(X)') \\
&= \inf R\,\underline{\mathrm{Hom}}_{A^{\mathrm{opp}}}(K[n], R\Gamma_{\mathfrak{m}}(X)') \\
&\geq n + \inf\,(R\Gamma_{\mathfrak{m}}(X)') \\
&= n - \sup R\Gamma_{\mathfrak{m}}(X) \\
&= (*).
\end{aligned}$$

Here $n = \sup R\Gamma_{\mathfrak{m}}(A)$, and by using the observation from before, that each complex X in $D^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{GrMod}(A))$ has $\sup R\Gamma_{\mathfrak{m}}(X) = \mathrm{GK}\dim(X)$, we get

$$(*) = \mathrm{GK}\dim(A) - \mathrm{GK}\dim(X).$$

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1): It is clear that if either of conditions (2) and (3) is satisfied, we have $\mathrm{depth}_A(A) \geq \mathrm{GK}\dim(A)$, that is

$$\inf R\Gamma_{\mathfrak{m}}(A) \geq \mathrm{GK}\dim(A) = \mathrm{K}\dim(A),$$

where we use [8, Lemma 6.1]'s statement that $\mathrm{GK}\dim(A) = \mathrm{K}\dim(A)$. On the other hand, by [1, Theorem 8.13], we have $\mathrm{lcd}(A) \leq \mathrm{K}\dim(A)$, so

$$\sup R\Gamma_{\mathfrak{m}}(A) \leq \mathrm{K}\dim(A).$$

So the complex $R\Gamma_{\mathfrak{m}}(A)$ is concentrated in one degree, and so, A is AS-Cohen–Macaulay. $\square$

Acknowledgements

The results proved above build on Van den Bergh's beautiful results from [9], which provides precise conditions characterizing algebras with a balanced dualizing complex. I would like to thank him for communicating his preprint.

References

[1] M. Artin, J.J. Zhang, Noncommutative projective schemes, Adv. Math. 109 (1994) 228–287.
[2] H.-B. Foxby, Gorenstein modules and related modules, Math. Scand. 31 (1972) 267–284.
[3] K.R. Goodearl, R.B. Warfield, An Introduction to Noncommutative Noetherian Rings, London Math. Soc. Stud. Texts, vol. 16, Cambridge University Press, Cambridge, 1989.
[4] P. Jørgensen, Local cohomology for non-commutative graded algebras, Comm. Algebra 25 (1997) 575–591.
[5] H. Matsumura, Commutative Ring Theory, Cambridge Stud. Adv. Math., vol. 8, Cambridge University Press, Cambridge, 1986.
[6] I. Reiten, The converse to a theorem of Sharp on Gorenstein modules, Proc. Amer. Math. Soc. 32 (1972) 417–420.
[7] R.Y. Sharp, Necessary conditions for the existence of dualizing complexes in commutative algebra, in: Séminaire Algèbre, P. Dubreil 1977/78, Lecture Notes in Math., vol. 740, Springer, Berlin, 1979, pp. 213–229.
[8] J.T. Stafford, J.J. Zhang, Homological Properties of (Graded) Noetherian PI Rings, J. Algebra 168 (1994) 988–1026.
[9] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered rings, J. Algebra 195 (1997) 662–679.
[10] A. Yekutieli, Dualizing complexes over noncommutative graded algebras, J. Algebra 153 (1992) 41–84.
[11] A. Yekutieli, J.J. Zhang, Serre duality for noncommutative projective schemes, Proc. Amer. Math. Soc. 125 (1997) 697–707.
[12] J.J. Zhang, Graded Gorenstein rings with enough normal elements, J. Algebra 189 (1997) 390–405.